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FOURIER-MOTZKIN ELIMINATION AND ITS DUAL

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Research on linear inequalities systems prior to 1947 consisted of isolated efforts by a few investigators. A case in point is the elimination technique for reducing the number of variables in the system. A description of the method can be found in Motzkin's 1936 Ph.D. thesis. It differs from its analog for systems of equations in that (unfortunately) each step in the elimination can greatly increase the number of inequalities in the remaining variables. For years the method was referred to as the Motzkin Elimination Method. However, because of the odd grave-digging custom of looking for artifacts in long forgotten papers, it is now known as the Fourier-Motzkin Elimination Method. In this paper we review the elimination scheme and show that a dual form of the method is a technique for reducing the number of equations in a system of equations in non-negative variables. Some comments regarding its appliability to integer programs also made.

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III

FOURIER-MOTZKIN ELIMINATION AND ITS DUAL

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Research on linear inequalities systems prior to 1947 consisted of isolated efforts by a few investigators. A case in point is the elimination technique for reducing the number of variables in the system. A description of the method can be found in Motzkin's 1936 Ph.D. thesis.[1] It differs from its analog for systems of equations in that (unfortunately) each step in the elimination can greatly increase the number of inequalities in the remaining variables. For years the method was referred to as the Motzkin Elimination Method. However, because of the odd grave-digging custom of looking for artifacts in long forgotten papers, it is now known as the Fourier-Motzkin Elimination Method.[2]

Given a system of linear inequalities: Find $x = (x_1, \ldots, x_n)$ such that

$$(1) \qquad \sum_{j=1}^{n} a_{ij} x_j \geq b_i , \qquad i = (1, \ldots, m) .$$

One may partition it into three sets of inequalities according to whether the coefficients of x_1 are positive, negative or zero. This permits rewriting (1) in the form:

[1]Motzkin, T. S., 1936, Doctoral Thesis, University of Basal, Beitrage zur theorie der Linearen Ungleichungen.

[2]Fourier, J. B. J., "Solution d'une question particuliere du calcul des inegalites," 1826 and extracts from "HIstoire de 'lAcademie", 1823, 1824, Oeuvres II, pp. 317-328 (French Academy of Sciences).

$$(2) \qquad \begin{cases} x_1 \geq D_1(\bar{x}) \\ \vdots \\ x_1 \geq D_p(\bar{x}) \end{cases} \quad \begin{cases} x_1 \leq E_1(\bar{x}) \\ \vdots \\ x_1 \leq E_q(\bar{x}) \end{cases} \quad \begin{cases} 0 \leq F_1(\bar{x}) \\ \vdots \\ 0 \leq F_r(\bar{x}) \end{cases}$$

where $D_i(\bar{x})$, $E_j(\bar{x})$, $F_k(\bar{x})$ are linear functions of $\bar{x} = (x_2, \ldots, x_n)$. It may be solved by first solving the <u>reduced</u> system: Find $\bar{x}$ satisfying

$$(3) \qquad \begin{array}{ll} D_i(\bar{x}) \leq E_j(\bar{x}) & i = (1,\ldots,p);\ j = (1,\ldots,q),\ k = (1,\ldots,r) \\ 0 \leq F_k(\bar{x}) & \end{array}$$

and then finding an x_1, satisfying

$$(4) \qquad \max_i D_i(\bar{x}) \leq x_1 \leq \min_j E_j(\bar{x}) \quad ,$$

where x_1 always exists providing there exists an $\bar{x}$ satisfying (3).

<u>Proof</u>: Given any $(x_1, \bar{x})$ satisfying (2), it is clear that (3) and (4) must hold. Conversely, given any $\bar{x}$ satisfying (3), then $\text{Max } D_i(\bar{x}) \leq \text{Min } E_j(\bar{x})$ and we can always find an x_1 satisfying (4); hence $(x_1, \bar{x})$ satisfies (1).

System (3) is said to be the result of "eliminating" x_1 from system (2). If $p+q \leq 4$, the reduced system contains one less variable and no more inequalities. If $p > 2$, $q > 2$, $r = 0$, however, the process of elimination will greatly increase the number of inequalities. This is the chief reason given why it is not used as a practical solution

method. It is worth noting, however, that (3) has special structure and that this might be used to advantage to develop it into a practical computational procedure.

Since (3) is a linear inequality system also, one could next proceed to eliminate x_2 etc. until one has eliminated all but a single variable, say x_n. The original system is solveable if and only if the final system $x_n \leq \alpha_i$, $x_n \geq \beta_j$, $0 \leq \gamma_k$ for $i = 1, \ldots, p'$, $j = 1, \ldots, q'$, $k = 1, \ldots, r'$ is consistent, i.e., iff $\alpha_i - \beta_j \geq 0$ and $\gamma_k \geq 0$ for all i,j,k. Another way to state this is

<u>Feasibility Theorem</u>: A necessary and sufficient condition that system (1) is solveable, is there exist no set of weights $(y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0)$ such that

$$(5) \qquad \sum_{i=1}^{m} y_i b_i > 0 \quad \text{and} \quad \sum_{i=1}^{m} y_i a_{ij} = 0 \qquad \text{for } j = (1,\ldots,n).$$

<u>Proof</u> (Abadie): Assume a solution x to (1) exists and there exists weights $y_i \geq 0$ satisfying (5), then (1) implies

$$(6) \qquad \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j \geq \sum_{i=1}^{m} y_i b_i \,, \qquad y_i \geq 0 \,,$$

or $0x \geq \sum y_i b_i > 0$, a contradiction. Thus the condition is necessary.

Assume no solution x to (1) exists, then note each system generated by the elimination process, for example (3) from (2), is formed

by non-negative linear combinations of the inequalities of the previous system which in turn were formed by non-negative linear combinations of the system one before that, etc., back to the original system (1). Thus the condition for non-solveability, $\alpha_i - \beta_j < 0$ or $\gamma_k < 0$ for some i,j or k (referred to earlier) could be derived directly by some non-negative linear combination of the inequalities of the original system.

This remarkably simple proof of the feasibility theorem based on Fourier-Motzkin elimination is due to Jean Abadie. From it one can derive easily (by trivial algebraic manipulations) the fundamental Duality Theorem of linear programming, Farkas Lemma, the various theorems of the alternatives, and the well known

Motzkin Transposition Theorem: Given the dual homogeneous linear program in partitioned form

(7) Primal: $A_I x_I + A_{II} x_{II} = 0$, $(x_I, x_{II}) \geq 0$

Dual: $yA_I \leq 0$, $yA_{II} \leq 0$,

then either there exists a solution to the dual such that $yA_I < 0$ (i.e., holds strictly in all components) or there exists a solution to the primal such that $x_I \neq 0$.

Proof: A solution to the dual such that $yA_I < 0$ implies there exists a y such that

$$(8) \qquad yA_I \leq -e , \qquad e = (1, 1, \ldots , 1)$$

$$yA_{II} \leq 0 .$$

If no such y exists satisfying (8), then by the feasibility theorem, there exists weights $x_I \geq 0$, $x_{II} \geq 0$ such that $A_I x_I + A_{II} x_{II} = 0$ and $-ex_I < 0$, i.e., $x_I \neq 0$.

The Dual of Fourier-Motzkin Elimination. Suppose we are given the homogeneous linear program

$$(9) \qquad \begin{cases} x_1 - D_i\bar{x} \geq 0 & i = (1, \ldots , p) \\ -x_1 + E_j\bar{x} \geq 0 & j = (1, \ldots , q) \\ F_k\bar{x} \geq 0 & k = (1, \ldots , r) \end{cases}$$

where $\bar{x} = (x_2, \ldots , x_n)$ and D_i, E_j, F_k are $1 \times n$. The elimination of x_1 from (9) yields

$$(10) \qquad (E_j - D_i)\bar{x} \geq 0 \qquad \text{for all } i,j$$

$$F_k\bar{x} \geq 0 \qquad \text{for all } k .$$

On the other hand the homogeneous dual of (9) is: To find $u_i \geq 0$, $v_j \geq 0$, $w_k \geq 0$ such that

$$(11)\quad \begin{aligned} &(a) \quad \sum_{i=1}^{p} u_i - \sum_{j=1}^{q} v_j = 0 \\ &(b) \quad -\sum_{i=1}^{p} u_i D_i + \sum_{j=1}^{q} v_j E_j + \sum_{k=1}^{r} w_k F_k = 0 \end{aligned}$$

and the homogeneous dual of (10) is: To find $\lambda_{ij} \geq 0$, $w_k \geq 0$ such that:

$$(12)\quad \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_{ij}(E_j - D_i) + \sum_{k=1}^{r} w_k F_k = 0 \ .$$

Since (9) and its eliminated form (10) are in a sense equivalent systems, it seems natural to expect that their duals (11) and (12), are also equivalent in the same sense; i.e., from any solution to (11) we can derive a solution to (12) and conversely. Note that (11) has n equations corresponding the the n components of x_j whereas (12) has $n-1$ equations but would have (in general) far more variables. This suggests we have at hand a technique for reducing the number of equations in a linear program. Let us give a direct proof of this for the non-homogeneous system:

Find $u_i \geq 0$, $v_j \geq 0$, $w_k \geq 0$ satisfying:

$$(13)\quad \begin{aligned} &(a) \quad \sum_{i=1}^{p} u_i - \sum_{j=1}^{q} v_j = 0 \ , \\ &(b) \quad -\sum_{i=1}^{p} u_i D_i + \sum_{j=1}^{q} v_j E_j + \sum_{k=1}^{r} w_k F_k = g \ . \end{aligned}$$

Let us introduce pq new variables $\lambda_{ij} \geq 0$ by setting

$$(14) \qquad \begin{aligned} u_i &= \sum_{j=1}^{q} \lambda_{ij} , & i &= (1, \ldots , p) \\ v_j &= \sum_{i=1}^{p} \lambda_{ij} , & j &= (1, \ldots , q) \end{aligned}$$

Note that if u_i and v_j satisfy (13)(a), it is always easy to find $u_{ij} \geq 0$ satisfying (14). Even if $u_i \geq 0$ and $v_j \geq 0$ are constrained to be <u>integers</u>, it is easy to find <u>integer</u> $\lambda_{ij} \geq 0$ satisfying (14). Substituting (14) into (13) we note that (13)(a) is automatically satisfied and we obtain the reduced system:

Find $\lambda_{ij} \geq 0$, $w_k \geq 0$ such that

$$(15) \qquad \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_{ij}(E_j - D_i) + \sum_{k=1}^{r} w_k F_k = g .$$

Conversely note that if we have a solution to (15), we can by regrouping the terms and substituting u_i and v_j for the resulting expression λ_{ij}, obtain a solution to (13). The solution will be in integers if λ_{ij} is integral.

To apply the technique to a system of equations in non-negative variables, it is necessary to have one equation with a zero constant term to play the role of (13)(a) or to create an equation with a zero constant term by replacing one of the equations by some appropriate linear combination of the equations of the system. This will yield an equation of the form

(16) $$\sum_{i=1}^{p} \alpha_i u_i - \sum_{j=1}^{q} b_j v_j = 0 \qquad \alpha_i \geq 0,\ \beta_j \geq 0\ ,$$

and we could obtain a system of form (13) by a change of units. This may conveniently be done by replacing (14) by

(17) $$\begin{aligned} \alpha_i u_i &= \sum_{j=1}^{q} \lambda_{ij}\ , & i &= (1, \ldots, p)\ , \\ \beta_j v_j &= \sum_{i=1}^{p} \lambda_{ij}\ , & j &= (1, \ldots, q)\ , \end{aligned}$$

where $\alpha_i \geq 0$, $\beta_j \geq 0$, $\lambda_{ij} \geq 0$, $u_i \geq 0$, $v_j \geq 0$.

<u>Application of the Dual of the Motzkin Elimination to Integer Programs</u>: As long as $\alpha_i = 1$, $\beta_j = 1$ for all i,j we have, as pointed out earlier, a reduced system of equations (16) in integer variables $\lambda_{ij} \geq 0$ if u_i and v_j are integers. In general, however, for the case where $\alpha_i > 0$ and $\beta_j > 0$ are integers different from unity, we have to resort to more complicated substitutions. This will be illustrated below for a simple example. Suppose we have

(18) $$(u_1 + 2u_2) - (v_1 + v_2 + v_3) = 0 \quad .$$

Let us rewrite this

(19) $$(u_1 + u_2 + u_3) - (v_1 + v_2 + v_3) = 0\ ,$$

where $u_2 = u_3$ and set as above

$$u_i = \sum_{j=1}^{3} \lambda_{ij} , \qquad j = (1, 2, 3)$$

(20)

$$v_j = \sum_{i=1}^{r} \lambda_{ij} , \qquad i = (1, 2, 3)$$

The resulting integer reduced system is in $\lambda_{ij} \geq 0$ (as before) except we have the additional condition $u_2 = u_3$ which, in terms of λ_{ij}, becomes

(21) $$(\lambda_{21} + \lambda_{22} + \lambda_{23}) - (\lambda_{31} + \lambda_{32} + \lambda_{33}) = 0$$

But (21) is in exactly the form we need for the integer reduction. We accordingly can introduce additional integer variables $\mu_{ij} \geq 0$, where

$$\lambda_{2i} = \sum_{j=1}^{3} \mu_{ij} , \qquad i = 1, 2, 3$$

(22)

$$\lambda_{3j} = \sum_{i=1}^{3} \mu_{ij} , \qquad j = 1, 2, 3$$

Back substituting into (20), we have the desired integer substitution in terms of 12 auxilliary variables.

$$u_1 = \sum_{j=1}^{3} \lambda_{1j}\ , \qquad u_2 (= u_3) = \sum_{i=1}^{3} \sum_{j=1}^{3} \mu_{ij}\ ,$$

$$(23)\qquad \begin{aligned} v_1 &= \lambda_{11} + \sum_{j=1}^{3} \mu_{1j} + \sum_{i=1}^{3} \mu_{i1} \\ v_2 &= \lambda_{12} + \sum_{j=1}^{3} \mu_{2j} + \sum_{i=1}^{3} \mu_{i2} \\ v_3 &= \lambda_{13} + \sum_{j=1}^{3} \mu_{3j} + \sum_{i=1}^{3} \mu_{i3} \end{aligned}$$

By setting $\mu_{12} + \mu_{21} = \bar{\mu}_{12}$, $\mu_{13} + \mu_{31} = \bar{\mu}_{13}$, $\mu_{32} + \mu_{23} = \bar{\mu}_{23}$ we could simplify the above substitution to one involving nine non-negative integer variables λ_{1i}, μ_{ii}, $\bar{\mu}_{ij}$ where $i,j = 1, 2, 3$ and $i \neq j$.

The problem in general of finding substitutions to replace (17) so as the reduce a linear system in non-negative integer variables to fewer equations is under study and will be the subject of a subsequent paper.